

Certain integrable system on a space associated with a quantum search algorithm

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On thinking of a Grover-type quantum search algorithm for an ordered tuple of multi-qubit states, a gradient system associated with the negative von-Neumann entropy is studied on the space of regular relative-configurations of multi-qubit states (SR^2CMQ). The SR^2CMQ emerges, through a geometric procedure, from the space of ordered tuples of multi-qubit states for the quantum search. The aim of this paper is to give a brief report on the integrability of the gradient dynamical system together with quantum information geometry of the underlying space, SR^2CMQ , of that system.

I. INTRODUCTION

Quantum computing has been investigated as one of the most challenging research subjects [1] over a decade. In 2001, Miyake and Wadati [2] provided a differential geometric characterization of the Grover search algorithm for a single target state [3] from a geometric viewpoint: They applied the fiber bundle structure of the space of normalized multi-qubit states over the complex projective space to the sequence of states generated by the search algorithm. The sequence after projection is shown to be along a geodesic of the complex

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projective space endowed with the Fubini-Study metric.

The aim of this paper is to report very briefly on a integrable dynamical system arising from geometric studies on a quantum search for an ordered tuple of multi-qubit states, which possesses rich quantum information features: The space for the integrable system is thought of as a ‘quantum information space’ since it is represented mathematically as the space of positive definite Hermitean matrices with unit trace (*i.e.*, regular density matrices) endowed with the symmetric logarithmic derivative (SLD) quantum Fisher metric. Further, the integrable system to be dealt with is the gradient system with the negative von-Neumann entropy as the potential. The quantum information space structure presented in this paper emerges in a purely geometric way motivated by Miyake and Wadati [2, 4]. The quantum information space structure with the SLD quantum Fisher metric is shown to be equivalent to the Riemannian structure arising from the fibered space structure over the space of regular ‘relative configurations’ of multi-qubit states in ordered tuples. The organization of this paper is outlined in what follows.

Section II is a preliminary section. The Hilbert space for a Grover-type search algorithm for an ordered tuple of multi-qubit states is presented together with a brief description of the search algorithm. The space of normalized ordered tuples of multi-qubit states where the search is proceeded will be abbreviated to STMQ.

In Section III, the space of regular ‘relative-configurations’ of multi-qubit states in ordered, tuples abbreviated to SR^2CMQ , is studied from geometric viewpoint. As the first step, the quotient space of STMQ under the left $U(2^n)$ action is introduced as the space of ‘relative-configurations’ of multi-qubit states in ordered tuple, abbreviated to SRCMQ. As the regular part of the SRCMQ free from singularities, SR^2CMQ is introduced and is shown to be isomorphic to the space of positive definite Hermitean matrices with unit trace. A pair of geometric structures are introduced to SR^2MQ : One is the Riemannian metric arising from the fibered space structure over SR^2CMQ . The other is the SLD (symmetric logarithmic derivative) quantum Fisher metric, on looking SR^2MQ mathematically upon as the space of non-singular density matrices. As one of the goals of this paper, the SLD quantum Fisher metric is shown to be identical to the Riemannian metric given above up to a multiplication constant.

Since SR^2MQ is shown to be isomorphic to the space of non-singular density matrices endowed with the SLD quantum Fisher metric in Sec. III and since the von-Neumann entropy

is known to be a typical quantum-information object on SR^2MQ , it would be very interesting to study the gradient system associated with the von-Neumann entropy in Section IV. On using a geometric calculus, the equation of motion for the gradient system is solved, so that the gradient system is shown to be integrable.

Section V is for concluding remarks. A pair of papers [5, 6] are in preparation which deal with the geometric study on SR^2MQ and the gradient system on SR^2MQ in more detail.

II. PRELIMINARIES

As is known very well, the Hilbert space for multi-qubit systems is the tensor product, $\overbrace{(\mathbf{C}^2) \otimes \cdots \otimes (\mathbf{C}^2)}^n \cong (\mathbf{C}^2)^{\otimes n}$, of the single-qubit space, \mathbf{C}^2 [1]. We prepare the direct sum,

$$\overbrace{(\mathbf{C}^2)^{\otimes n} \oplus \cdots \oplus (\mathbf{C}^2)^{\otimes n}}^m, \quad (1)$$

of the Hilbert space $(\mathbf{C}^2)^{\otimes n}$ for n -qubit states to describe ordered tuples of multi-qubit states, so that any ordered tuple of n -qubit states is expressed as

$$\Phi = (\phi^{(1)}, \dots, \phi^{(m)}) \quad (\phi^{(j)} \in (\mathbf{C}^2)^{\otimes n}, j = 1, \dots, m), \quad (2)$$

where m indicates the number of n -qubit states in ordered tuples. On writing out each $\phi^{(j)} \in (\mathbf{C}^2)^{\otimes n}$ in Φ of (2) to be the column-vector form,

$$\phi^{(j)} = (\phi_1^{(j)}, \phi_2^{(j)}, \dots, \phi_{2^n}^{(j)})^T \in (\mathbf{C}^2)^{\otimes n} \quad (j = 1, 2, \dots, m; \text{ } ^T \text{ the transpose}), \quad (3)$$

Φ of (2) has the complex $2^n \times m$ matrix form with the components

$$\Phi_{h\ell} = \phi_h^{(\ell)} \quad (h = 0, \dots, 2^n - 1; \ell = 1, \dots, m). \quad (4)$$

We can thereby identify the space of ordered tuples of n -qubit states with the space of complex $2^n \times m$ matrices denoted by $M(2^n, m; \mathbf{C})$ henceforth. The $M(2^n, m; \mathbf{C})$ admits the conventional Hilbert-space structure associated with the Hermitean inner product,

$$\langle \Phi, \Phi' \rangle = \frac{1}{m} \text{tr} (\Phi^\dagger \Phi) \quad (\Phi, \Phi' \in M(2^n, m; \mathbf{C})). \quad (5)$$

As the computational basis, the set of matrices, $\Phi(j; k)$ ($j = 1, \dots, m, k = 0, \dots, 2^n - 1$), with the components

$$(\Phi(j; k))_{h\ell} = \sqrt{m} \delta_{jh} \delta_{k\ell} \quad (h = 0, \dots, 2^n - 1; \ell = 1, \dots, m) \quad (6)$$

is taken, where δ_{jh} and $\delta_{k\ell}$ indicate the Kronecker delta. The subset,

$$M_1(2^n, m; \mathbf{C}) = \left\{ \Phi \in M(2^n, m; \mathbf{C}) \mid \langle \Phi, \Phi \rangle = \frac{1}{m} \text{tr}(\Phi^\dagger \Phi) = 1 \right\}, \quad (7)$$

of $M(2^n, m; \mathbf{C})$ is taken as the space of ‘quantum states’, which is diffeomorphic to the unit sphere of dimension $2^{n+1}m - 1$. The space $M_1(2^n, m; \mathbf{C})$ represents the space of normalized ordered tuples of multi-qubit states, STMQ.

The Grover-type search algorithm for an ordered tuple of multi-qubit states is described in the following. Let us start with defining the initial tuple to be the sum of all the computational-base states with equal weight;

$$A = \frac{1}{\sqrt{2^n m}} \sum_{j=1}^m \sum_{k=0}^{2^n-1} \Phi(j; k) \in M_1(2^n, m; \mathbf{C}). \quad (8)$$

The target tuple denoted by $W \in M_1(2^n, m; \mathbf{C})$ is assumed to be the tuple of distinct computational base-states;

$$W = \frac{1}{\sqrt{m}}(w^{(1)}, w^{(2)}, \dots, w^{(m)}), \quad w^{(h)} = \Phi(\exists j_h; \exists k_h) \quad \text{with} \quad w^{(h)} \neq w^{(h')} \quad (h \neq h'). \quad (9)$$

Like the original case due to Grover [3], the search process is made by applying the composition $-I_A \circ I_W$ of unitary operators,

$$I_W(\Phi) = \Phi - 2\langle W, \Phi \rangle W \quad \text{and} \quad I_A(\Phi) = \Phi - 2\langle A, \Phi \rangle A \quad (\Phi \in M(2^n, m; \mathbf{C})), \quad (10)$$

repeatedly to the initial tuple A . By calculation, we have

$$(-I_A \circ I_W)^k(A) = \left(\cos(k + \frac{1}{2})\theta \right) R + \left(\sin(k + \frac{1}{2})\theta \right) W \quad (k = 0, 1, 2, \dots) \quad (11)$$

with $R = \sqrt{2^n/(2^n - 1)}A - \sqrt{1/(2^n - 1)}W$ and $\sin(\theta/2) = \sqrt{1/2^n}$ ($\theta \in (0, \pi)$). In the case that the degree of qubit, n , is large enough, the probability of finding W in $(-I_A \circ I_W)^k A$ gets closed to one as k does to $(\pi\sqrt{2^n} - 1)/2$.

III. THE GEOMETRY OF SR²MQ

In this section, we study the space, SR²CMQ, of regular ‘relative-configurations’ of multi-qubit states in ordered tuples both from the fibered space structure and the quantum-information structure viewpoints.

A. Setting-up: SRCMQ and SR²CMQ

We start with giving a description of the relative-configuration of multi-qubit states in ordered tuples. Let us take $\Phi \in M_1(2^n, m; \mathbf{C})$ all of whose column vectors, $\phi^{(j)}$ s, in (2) are non-vanishing [7]. By the placement of the column vectors $\phi^{(k)}$ ($k > 1$) relative to $\phi^{(1)}$, we mean the relative-configuration of multi-qubit states in that ordered tuple Φ . For $\Phi \in M_1(2^n, m; \mathbf{C})$ with non-vanishing columns, let us consider the matrix $g\Phi \in M_1(2^n, m; \mathbf{C})$ with $g \in U(2^n)$, each of whose column vectors, $g\phi^{(j)}$ s, are given by the unitary action of g to $\phi^{(j)}$ s by g . Then we can say that the relative-configurations of multi-qubit states in Φ and those in $g\Phi$ are the same. Accordingly, as the space, SRCMQ, of relative-configurations of multi-qubit states, the quotient space $M_1(2^n, m; \mathbf{C})/\sim$ of $M_1(2^n, m; \mathbf{C})$ is thought of, which is associated with the equivalence relation

$$\Phi \sim \Phi' \quad \text{if and only if} \quad \exists g \in U(2^n) \quad \text{s.t.} \quad \Phi' = g\Phi, \quad (12)$$

generated by the $U(2^n)$ action on $M_1(2^n, m; \mathbf{C})$. For $M_1(2^n, m; \mathbf{C})/\sim$, we have the following.

Proposition 1 *The quotient space $M_1(2^n, m; \mathbf{C})/\sim$ is isomorphic the space of positive semi-definite $m \times m$ Hermitean matrices with unit trace,*

$$H_{\geq 0,1}^m = \{\rho \in M(m, m; \mathbf{C}) \mid \rho^\dagger = \rho, \text{tr } \rho = 1, \rho : \text{positive semi-definite}\}. \quad (13)$$

The space, SRCMQ, of relative configurations of multi-qubit states in ordered tuples will be represented in the form $H_{\geq 0,1}^m$, henceforth.

The proof is accomplished by showing that the map,

$$\pi_m : \Phi \in M_1(2^n, m; \mathbf{C}) \mapsto \frac{1}{m} \Phi^\dagger \Phi \in H_{\geq 0,1}^m, \quad (14)$$

is surjective and that $\pi_m(\Phi) = \pi_m(\Phi')$ holds true iff so does (12) [9].

We wish to specify a ‘regular’ part of SRCMQ on which we can make differential calculus without any problems. In view of (13), $H_{\geq 0,1}^m$ (SRCMQ) has a natural boundary, $\partial H_{\geq 0,1}^m = \{\rho \in H_{\geq 0,1}^m \mid \det \rho = 0\}$, so that we have

$$H_{+,1}^m := H_{\geq 0,1}^m - \partial H_{\geq 0,1}^m = \{\rho \in M(m, m; \mathbf{C}) \mid \rho^\dagger = \rho, \det \rho > 0, \text{tr } \rho = 1\}. \quad (15)$$

The $H_{+,1}^m$ is nothing but the set of positive definite $m \times m$ Hermitean matrices with unit trace. We have the following proposition providing a differentiable structure to $H_{+,1}^m$.

Proposition 2 *The inverse image, $\pi_m^{-1}(H_{+,1}^m)$ of $H_{+,1}^m$ by π_m is made into the fiber bundle, $\pi_m : \pi_m^{-1}(H_{+,1}^m) \rightarrow H_{+,1}^m$, with the fiber diffeomorphic to $U(2^n)/U(2^n - m)$.*

We here present only a key to the proof: A key is that the isotropy subgroup, $G_\Phi = \{g \in U(2^n) \mid L_g(\Phi) = g\Phi = \Phi\}$, of $U(2^n)$ at any $\Phi \in \pi_m^{-1}(H_{+,1}^m)$ is isomorphic to $U(2^n - m)$ [5]. According to transformation-group theory [10], this fact implies that any $\Phi \in \pi_m^{-1}(H_{+,1}^m)$ has the same orbit type in common. This proves our assertion.

We wish to give an account of referring to $H_{+,1}^m$ as SR²CMQ. From (13), it is easy to see that any $\Phi \in \pi_m^{-1}(H_{+,1}^m)$ is of rank m . This implies the linear independence among the column vectors of any $\Phi \in \pi_m^{-1}(H_{+,1}^m)$, while the linear independence does not hold true for any $\Phi \in \pi_m^{-1}(\partial H_{\geq 0,1}^m)$. We may thereby think any $\Phi \in \pi_m^{-1}(H_{+,1}^m)$ is endowed with a sort of regularity in the relative-configuration. We reach to the following.

Definition 3 *The space, SR²CMQ, of regular relative-configurations of multi-qubit states in ordered tuples is represented as the space of positive definite Hermitean matrices with unit trace $H_{+,1}^m$.*

B. The fibered space structure and the Riemannian metric

We are to give the Riemannian metric of $H_{+,1}^m$, namely SR²CMQ, that makes the projection, $\pi_m : \pi_m^{-1}(H_{+,1}^m) \rightarrow H_{+,1}^m$, the Riemannian submersion. We start with the Riemannian metric of $\pi_m^{-1}(H_{+,1}^m)$. Recalling the fact that $\pi_m^{-1}(H_{+,1}^m)$ is open in $M_1(2^n, m; \mathbf{C})$ together with the definition (7) of $M_1(2^n, m; \mathbf{C})$, the tangent space of $\pi_m^{-1}(H_{+,1}^m)$ at Φ is given by

$$T_\Phi \pi_m^{-1}(H_{+,1}^m) = \{X \in M(2^n, m; \mathbf{C}) \mid \Re(\text{tr}(\Phi^\dagger X)) = 0\} \quad (\Phi \in \pi_m^{-1}(H_{+,1}^m)). \quad (16)$$

From the Hermitean inner product (5) of $M(2^n, m; \mathbf{C})$ ($\supset \pi_m^{-1}(H_{+,1}^m)$), the Riemannian metric denoted by (\cdot, \cdot) arises, which endows the inner product

$$(X, X')_\Phi = \frac{1}{m} \Re(\text{tr}(X^\dagger X')) \quad (X, X' \in T_\Phi \pi_m^{-1}(H_{+,1}^m)) \quad (17)$$

to any $T_\Phi \pi_m^{-1}(H_{+,1}^m)$. Using the metric (\cdot, \cdot) of $\pi_m^{-1}(H_{+,1}^m)$, we wish to give the orthogonal direct-sum decomposition,

$$T_\Phi \pi_m^{-1}(H_{+,1}^m) = \ker(\pi_{m*,\Phi}) \oplus (\ker(\pi_{m*,\Phi}))^\perp, \quad (18)$$

explicitly, where $\ker(\pi_{m*}, \Phi)$ denotes the kernel of the tangent map,

$$(\pi_m)_{*, \Phi} : X \in T_\Phi \pi_m^{-1}(H_{+,1}^m) \mapsto \frac{1}{m} (\Phi^\dagger X + X^\dagger \Phi) \in T_{\pi_m(\Phi)} H_{+,1}^m \quad (\Phi \in \pi_m^{-1}(H_{+,1}^m)), \quad (19)$$

of π_m at Φ and $\ker(\pi_{m*}, \Phi)^\perp$ the orthogonal complement of the kernel with respect to $(\cdot, \cdot)_\Phi$.

Proposition 4 *Let a given $\Phi \in \pi_m^{-1}(H_{+,1}^m)$ admit the singular decomposition [8, 9],*

$$\begin{aligned} \Phi &= g \begin{pmatrix} \sqrt{m} \Lambda \\ O_{2^n-m, m} \end{pmatrix} h^\dagger \quad \text{with } g \in U(2^n), h \in U(m), \\ \Lambda &= \text{diag}(\lambda_1, \dots, \lambda_m) \quad \text{s.t.} \quad \sum_{j=1}^m \lambda_j^2 = 1, \quad \lambda_j > 0 \quad (j = 1, \dots, m). \end{aligned} \quad (20)$$

Then $\ker(\pi_{m*}, \Phi)$ and $\ker(\pi_{m*}, \Phi)^\perp$ take the following forms:

$$\begin{aligned} \ker(\pi_{m*}, \Phi) &= \left\{ X \in T_\Phi \pi_m^{-1}(H_{+,1}^m) \left| X = g \begin{pmatrix} (\sqrt{m} \Lambda)^{-1} \eta \\ x \end{pmatrix} h, \right. \right. \\ &\quad \left. \left. \eta \in u(m), x \in M(2^n - m, m; \mathbf{C}) \right\}, \quad (21) \\ \ker(\pi_{m*}, \Phi)^\perp &= \left\{ X \in T_\Phi \pi_m^{-1}(H_{+,1}^m) \left| X = g \begin{pmatrix} (\sqrt{m} \Lambda)^{-1} (\sigma + \alpha_\Lambda(\sigma)) \\ O_{2^n-m, m} \end{pmatrix} h, \right. \right. \\ &\quad \left. \left. \sigma \in M(m, m; \mathbf{C}), \sigma^\dagger = \sigma, \text{tr } \sigma = 0 \right\}, \quad (22) \end{aligned}$$

where $M(2^n - m, m; \mathbf{C})$ and $O_{2^n-m, m}$ denote the set of $(2^n - m) \times m$ complex matrices and the $(2^n - m) \times m$ null matrix, respectively. The α_Λ in (22) is the linear map of $m \times m$ traceless Hermitean matrices into the anti-Hermitean matrices which is defined to satisfy

$$\Lambda^{-2} \alpha_\Lambda(\sigma) + \alpha_\Lambda(\sigma) \Lambda^{-2} = -\Lambda^{-2} \sigma + \sigma \Lambda^{-2}, \quad (23)$$

where Λ appears in the singular-value decomposition (20).

We move on to define the horizontal lift of tangent vectors of $H_{+,1}^m$, where the tangent space $T_\rho H_{+,1}^m$ at $\rho \in H_{+,1}^m$ is given by the space of traceless $m \times m$ Hermitean matrices,

$$T_\rho H_{+,1}^m = \{\Xi \in M(m, m; \mathbf{C}) \mid \Xi^\dagger = \Xi, \text{tr } \Xi = 0\}. \quad (24)$$

Definition 5 For a given $\rho \in H_{+,1}^m$, let $\Phi \in \pi_m^{-1}(\rho)$ be chosen arbitrarily. Then, for any $\Xi \in T_\rho H_{+,1}^m$, there exists the unique tangent vector, $\ell_\Phi(\Xi) \in T_\Phi \pi_m^{-1}(H_{+,1}^m)$, at Φ subject to

$$(\pi_{m*})_\Phi(\ell_\Phi(\Xi)) = \Xi \quad \text{and} \quad \ell_\Phi(\Xi) \in \ker((\pi_{m*})_\Phi). \quad (25)$$

The tangent vector $\ell_\Phi(\Xi)$ at $\Phi \in \pi_m^{-1}(H_{+,1}^m)$ is called the horizontal lift of $\Xi \in T_\rho H_{+,1}^m$ to $T_\Phi \pi_m^{-1}(H_{+,1}^m)$.

The horizontal lift $\ell_\Phi(\Xi)$ is shown to take the following form [9].

Proposition 6 Let $\rho \in H_{+,1}^m$ be expressed in the form

$$\rho = h\Delta h^\dagger \text{ with } h \in U(m), \Delta = \text{diag}(\delta_1, \dots, \delta_m) \text{ s.t. } \sum_{j=1}^m \delta_j = 1, \delta_j > 0 (j = 1, \dots, m), \quad (26)$$

so that $\Phi \in \pi_m^{-1}(\rho)$ admits the singular-value decomposition (20) with $\lambda_j = \sqrt{\delta_j}$ ($j = 1, \dots, m$). The horizontal lift, $\ell_\Phi(\Xi)$, of $\Xi \in T_\rho H_{+,1}^m$ takes the form

$$\ell_\Phi(\Xi) = \frac{\sqrt{m}}{2} g \begin{pmatrix} \Delta^{-1/2}(h^\dagger \Xi h + \alpha_{\sqrt{\Delta}}(h^\dagger \Xi h)) \\ O_{2^n-m,m} \end{pmatrix} h^\dagger, \quad (27)$$

where $\sqrt{\Delta}$ stands for the square root of the diagonal matrix Δ and $\alpha_{\sqrt{\Delta}}$ is given by (23) with $\Lambda = \sqrt{\Delta}$.

In terms of the horizontal lift and the metric (\cdot, \cdot) of $\pi_m^{-1}(H_{+,1}^m)$, the Riemannian metric of $\pi_m^{-1}(H_{+,1}^m)$ that we are seeking is defined as follows.

Definition 7 The Riemannian metric, $((\cdot, \cdot))^R$, that makes $\pi_m : \pi_m^{-1}(H_{+,1}^m) \rightarrow H_{+,1}^m$ the Riemannian submersion is defined to provide the inner product subject to

$$((\Xi, \Xi'))_\rho^R = (\ell_\Phi(\Xi), \ell_\Phi(\Xi'))_\Phi \quad (\Xi, \Xi' \in T_\rho H_{+,1}^m, \rho \in H_{+,1}^m), \quad (28)$$

where $\Phi \in \pi_m^{-1}(\rho)$ is chosen arbitrarily.

Note that by the validity of (28) the π_m is meant to be the Riemannian submersion from $(\pi_m^{-1}(H_{+,1}^m), (\cdot, \cdot))$ to $(H_{+,1}^m, ((\cdot, \cdot))^R)$. A straightforward calculation shows the following.

Proposition 8 Let $\rho \in H_{+,1}^m$ is expressed in the form (26). The Riemannian metric $((\cdot, \cdot))^R$ of $H_{+,1}^m$ making π_m the Riemannian submersion takes the form,

$$((\Xi, \Xi'))_\rho^R = \frac{1}{2} \sum_{j,k=1}^m \frac{1}{\delta_j + \delta_k} \bar{\chi}_{jk} \chi'_{jk} \quad \text{with} \quad \chi = h^\dagger \Xi h, \chi' = h^\dagger \Xi' h. \quad (29)$$

C. The quantum information geometry

In the previous subsections, the space of positive semi-definite Hermitean matrices, $H_{\geq 0,1}^m$ is given as SRCMQ through the dimension-reduction of $M_1(2^n, m; \mathbf{C})$ by the $U(2^n)$ action. Since $H_{\geq 0,1}^m$ can be looked upon mathematically as the space of $m \times m$ density matrices, it is quite natural for us to come to study quantum information geometry of $H_{\geq 0,1}^m$ or of its open-dense subset $H_{+,1}^m$ representing SR^2CMQ . In the following, the SLD quantum Fisher metric is studied as a typical object, which is known to play a core role in quantum statistical theory on the space of density matrices [11].

We start with defining the symmetric logarithmic derivative (SLD). An account for the SLD is given below together with its classical counterpart: In classical theory, the Fisher metric (namely, the Fisher information matrix) is utilized to ‘measure’ the distance between a pair of points in the space of certain parametric probability distributions. The logarithmic derivative of distribution functions is defined to be the derivation of the likelihood functions for those distributions, which plays a crucial role in defining the classical Fisher metric. Accordingly, a quantum counterpart of the logarithmic derivation is required to define the quantum Fisher metric, which is however not determined uniquely due to the non-commutativity among matrices. As a conventional one, we introduce the symmetric logarithmic derivative (SLD) as follows [11].

Definition 9 *Let Ξ be any tangent vector at $\rho \in H_{+,1}^m$. The SLD, denoted by $\mathcal{L}_\rho(\Xi)$, for Ξ is defined to be the linear map that satisfies*

$$\frac{1}{2} \{ \rho \mathcal{L}_\rho(\Xi) + \mathcal{L}_\rho(\Xi) \rho \} = \Xi \quad (\Xi \in T_\rho H_{+,1}^m). \quad (30)$$

A straightforward calculation shows the following:

Proposition 10 *Let $\rho \in H_{+,1}^m$ is expressed in the form (26). With the notations*

$$\chi_{jk} = (h^\dagger \Xi h)_{jk}, \quad L_{jk} = (h^\dagger \mathcal{L}_\rho(\Xi) h)_{jk} \quad (j, k = 1, \dots, m), \quad (31)$$

the SLD, $\mathcal{L}_\rho(\Xi)$, for $\Xi \in T_\rho H_{+,1}^m$ takes the form

$$L_{jk} = \frac{2}{\delta_j + \delta_k} \chi_{jk} \quad (j, k = 1, \dots, m). \quad (32)$$

Note that the SLD cannot be extended to the whole $H_{\geq 0,1}^m$ due to (32) and to the fact that any $\rho \in \partial H_{\geq 0,1}^m$ admit the null-eigenvalue. The SLD quantum Fisher metric is defined as follows.

Definition 11 *The quantum SLD Fisher metric $((\cdot, \cdot))^{QF}$ is defined to give the inner product $((\cdot, \cdot))_\rho^{QF}$ in $T_\rho H_{+,1}^m$ subject to*

$$((\Xi, \Xi'))_\rho^{QF} = \frac{1}{2} \Re \left[\text{tr} \left[\rho (L_\rho(\Xi) L_\rho(\Xi') + L_\rho(\Xi') L_\rho(\Xi)) \right] \right] \quad (\Xi, \Xi' \in T_\rho H_{+,1}^m). \quad (33)$$

Equations (32) and (33) are combined together to yield an explicit expression of $((\Xi, \Xi'))_\rho^{QF}$, which is compared with (29) to show the first main theorem of this paper.

Theorem 12 (The 1st main theorem) *The quantum SLD Fisher metric $((\cdot, \cdot))^{QF}$ of the space, SR^2CMQ , of regular relative-configurations of multi-qubit states in ordered tuples coincides with the Riemannian metric $((\cdot, \cdot))^R$ up to the multiplication constant 4;*

$$((\Xi, \Xi'))_\rho^{QF} = 4((\Xi, \Xi'))_\rho^R \quad (\Xi, \Xi' \in T_\rho H_{+,1}^m, \rho \in H_{+,1}^m). \quad (34)$$

Thus the pair of geometric structures of the SR^2CMQ are shown to coincide with each other up to the multiplication constant. One is the Riemannian metric arising from the geometry of the fibered space structure over SR^2CMQ . The other is the SLD quantum Fisher metric arising from the (mathematical) identification of SR^2CMQ with the space of regular density matrices [12]. The coincidence seems to be accidental in the present circumstances though, since those metrics have been given rise from different concepts and then shown their coincidence by calculations. It would be interesting to seek a good account for connecting those objects.

IV. AN INTEGRABLE GRADIENT SYSTEM ON SR^2CMQ

This section is devoted to present an integrable gradient system on the SR^2CMQ , which has quantum information features.

A. The von-Neumann entropy on the SR^2CMQ

On regarding $H_{+,1}^m$ (SR^2CMQ) as an open-dense subspace of the space of density matrices, one of the most typical quantum-information objects on $H_{+,1}^m$ would be the von-Neumann entropy defined by

$$S(\rho) = -\text{tr} (\rho \log \rho) \quad (\rho \in H_{+,1}^m), \quad (35)$$

which can be extended to $H_{\geq 0,1}^m$. On solving the maximum-value problem of $S(\rho)$, the target state $W \in M_1(2^n, m; \mathbf{C})$ for our search is characterized as follows.

Proposition 13 *Any target tuple W given by (9) for the search is projected through $\pi_m : \pi_m^{-1}(H_{+,1}^m) \rightarrow H_{+,1}^m$ to the unique maximum point, $\frac{1}{m}I$, of $S(\rho)$, where I denotes the $m \times m$ identity matrix.*

Due to Prop. 13, the projection, $\{\pi_m((-I_A \circ I_W)^k(W))\}_{k=0,1,2,\dots}$, of the sequence generated by the search algorithm is understood to get close to the maximum point $(1/m)I$, of $S(\rho)$ if k gets close to $(\pi\sqrt{2^n} - 1)/2$. This fact inspires us to study the gradient dynamical system associated with $-S(\rho)$, all of whose trajectories besides the stationary one tend to $(1/m)I = \pi_m(W)$.

B. The equation of motion for the gradient system

As shown in the previous section, we have endowed the pair of non-Euclidean metrics to $H_{+,1}^m$, so that the gradient operator has to be associated with either of those. Since a quantum information aspect is one of our interest on this system, the SLD quantum Fisher metric $((\cdot, \cdot))^{QF}$ is chosen to describe the gradient operator [13]. Then the gradient vector of $-S(\rho)$ at ρ , denoted by $(\text{grad}_\rho(-S))(\rho)$, is defined to satisfy

$$((\text{grad}_\rho(-S))(\rho), \Xi)^{QF}_\rho = d(-S)_\rho(\Xi) \quad (\Xi \in T_\rho H_{+,1}^m), \quad (36)$$

where $d(-S)_\rho$ on the rhs of (36) denotes the exterior differential of $-S(\rho)$ at ρ . In a more intuitive way, the rhs can be written as the differentiation,

$$d(-S)_\rho(\Xi) = \left. \frac{d}{d\tau} \right|_{\tau=0} [-S(\gamma(\tau))] \quad (37)$$

along a smooth curve $\gamma(\tau)$ on $H_{+,1}^m$ subject to $\gamma(0) = \rho$ and $(d\gamma/d\tau)(0) = \Xi$ ($0 \leq |\tau| < \Xi_\varepsilon$). The equation of motion for the gradient system on the quantum information space $(H_{+,1}^m, ((\cdot, \cdot))^{QF})$ [12] with the potential $-S(\rho)$ is described by

$$\dot{\rho} = -(\text{grad}_\rho(-S))(\rho), \quad (38)$$

where $\dot{\cdot}$ indicates the differentiation by the time-parameter t henceforth.

We are to express the rhs of (38) more explicitly through a geometric way [6]. Let $\tilde{S}(\Phi)$ be the $U(2^n)$ invariant function

$$\tilde{S}(\Phi) = -\text{tr} \left[\frac{1}{m} \Phi^\dagger \Phi \log \left(\frac{1}{m} \Phi^\dagger \Phi \right) \right] \quad (39)$$

which satisfies $S \circ \pi_m = \tilde{S}$. Then the identities,

$$dS_\rho(\Xi) = d\tilde{S}_\Phi(\ell_\Phi(\Xi)) \quad \text{and} \quad ((\dot{\rho}, \Xi))_\rho^{QF} = 4(\ell_\Phi(\dot{\rho}), \ell_\Phi(\Xi))_\Phi \quad \begin{pmatrix} \Xi \in T_\rho H_{+,1}^m, \\ \Phi \in \pi_m^{-1}(H_{+,1}^m) \end{pmatrix}, \quad (40)$$

are put together to provide the explicit expression,

$$\dot{\rho} = \rho \left[(\text{tr}(\rho \log \rho)) I - \log \rho \right], \quad (41)$$

of (38), where ℓ_Φ denotes the horizontal-lift operation (25) and I the $m \times m$ identity matrix. The $\log \rho$ is the logarithm of the matrix ρ , which is well-defined for any $\rho \in H_{+,1}^m$.

C. Solution and integrability

We are to solve (41) under the ansatz

$$\rho(t) = h(t) \Delta(t) h(t)^\dagger \quad \text{with} \quad \Delta(t) = \text{diag}(\delta_1(t), \dots, \delta_m(t)), \quad h(t) \in U(m), \quad (42)$$

together with

$$h(t) \equiv h(0) \in U(m) \quad (t \in \mathbf{R}). \quad (43)$$

Under (42) and (43), Eq. (41) reduces to the system of differential equations,

$$\dot{\delta}_j = \delta_j(t) \left[\sum_{k=1}^m \delta_k(t) \log \delta_k(t) \right] - \delta_j(t) \log \delta_j(t) \quad (j = 1, \dots, m), \quad (44)$$

for δ_j s, whose initial value problem is solved to be

$$\delta_j(t) = \left(\sum_{k=1}^m (\delta_k(0))^{\exp(-t)} \right)^{-1} (\delta_j(0))^{\exp(-t)} \quad (j = 1, \dots, m). \quad (45)$$

Surprisingly, Eq. (44) takes exactly the same form as the equation of motion for the gradient system on the space of multinomial distributions with the negative entropy as the potential [14]. Combining (45) with the ansatz (42), we have the following.

Proposition 14 *Let the triple, $(H_{+,1}^m, ((\cdot, \cdot))^{QF}, -S(\rho))$, express the gradient system on the quantum information space $(H_{+,1}^m, ((\cdot, \cdot))^{QF})$ with the potential $-S(\rho)$, where $S(\rho)$ denotes the von-Neumann entropy defined by (35). The solution of the equation of motion (38) for $(H_{+,1}^m, ((\cdot, \cdot))^{QF}, -S(\rho))$ takes the form,*

$$\rho(t) = h(0) \left[\text{tr} \left[(\Delta(0))^{\exp(-t)} \right] \right]^{-1} (\Delta(0))^{\exp(-t)} h(0)^\dagger, \quad (46)$$

where $h(0) \in U(m)$ and $\Delta(t) = \text{diag}(\delta_1(0), \dots, \delta_m(0))$ are given by the initial condition, $\rho(0) = h(0)\Delta(0)h(0)^\dagger$ with (42).

Now that we have got the solution of the gradient system explicitly, we show the integrability of the gradient system $(H_{+,1}^m, ((\cdot, \cdot))^{QF}, -S(\rho))$ in turn. Note that the integrability here means the existence of $\dim H_{+,1}^m - 1$ constants of motions being mutually independent. We start with the construction of $m - 2$ independent constants of motion in δ_j s ($j = 1, \dots, m$). From (45) and (46), we find

$$\frac{\log \delta_{j+2}(t) - \log \delta_{j+1}(t)}{\log \delta_{j+1} - \log \delta_j(t)} \quad (j = 1, \dots, m - 2) \quad (47)$$

as constants of motion being mutually independent. Further from Proposition 14 with the expression (42), we find the unitary matrix $h(t)$ is kept invariant to be $h(0)$, from which we take $m(m - 3)/2$ independent constants of motion [15]. We have thus found mutually-independent $\dim H_{+,1}^m - 1$ ($= m(m + 1)/2 - 2$) constants of motion for the gradient system. To summarize, we reach to the second main theorem of this paper.

Theorem 15 (The 2nd main theorem) *The gradient system on the quantum information space $(H_{+,1}^m, ((\cdot, \cdot))^{QF})$ with the negative von Neumann entropy $-S(\rho)$ as the potential is integrable in the sense that it admits mutually-independent $\dim H_{+,1}^m - 1$ constants of motion.*

V. CONCLUDING REMARKS

We have made the geometric studies on SR^2CMQ , the space of regular relative-configurations of multi-qubit states in ordered tuples which comes from the space for the quantum search for an ordered tuple of multi-qubit states. We have shown that SR^2CMQ admits the Riemannian metric arising from the fibered space structure over SR^2CMQ and the SLD quantum Fisher metric, which turn out to be the same up to the multiplication constant.

On the quantum information space $(H_{+,1}^m, ((\cdot, \cdot))^{QF})$ [12], the gradient system with the negative von-Neumann entropy $-S(\rho)$ is taken to study, which is shown to be integrable. The equation (44) describing the time-evolution of spectra δ_j s of ρ turns out to be the same as that of the gradient system associated with multinomial distributions [14] appearing in the classical statistical theory, although we have to take the additional degree-of-freedom generated by $U(m)$ into account. Further, our integrable gradient system might be of new type. To study more on our gradient system, the Lax representation would be worth being investigated.

On closing this paper, the authors wish to mention of a future investigation: The integrability of our gradient system has inspired us to think of a possibility of a search algorithm with a kind of ‘convergence’ through a discretization of the gradient equation.

Acknowledgments

All the authors would like to thank Prof. T. Iwai and Prof. Y. Nakamura at Kyoto University for their valuable comments on this work.

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